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Pareto Optimality in Nonconvex Economies with Infinite-dimensional Commodity Spaces ⋆

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Abstract. The paper contains applications of variational analysis to the study of Pareto optimality in nonconvex economies with infinite-dimensional commodity spaces satisfying the Asplund property. Our basic tool is a certain extremal principle that provides necessary conditions for set extremality and can be treated as a variational extension of the classical convex separation principle to systems of nonconvex sets. In this way we obtain new versions of the generalized second welfare theorem for nonconvex economies in terms of appropriate normal cones of variational analysis.

Key words: Pareto optimality, Welfare economics, Variational analysis, Generalized differentiation, Extremal principle

1. Introduction

The primary goal of this paper is to study general concepts of Pareto optimality in nonconvex models of welfare economics with infinite-dimensional commodity spaces. To obtain new results in this direction, we employ powerful tools of modern variational analysis dealing with nonsmooth and nonconvex structures.

The classical Walrasian equilibrium model of welfare economics and its various generalizations have long been recognized as important part of the economic theory and applications. It has been well understood that the concept of Pareto optimality (efficiency) and its variants play a crucial role for the study of equilibria and making the best decisions for competitive economies.

A classical approach to the study of Pareto optimality in economic models with smooth data consists of reducing it to conventional problems of mathematical programming and using first-order *necessary optimality conditions* that involve Lagrange multipliers. In this way important results were obtained at the late 1930s and in the 1940s when it was shown that the marginal rates of substitution for consumption and production are equal to each other at any Pareto optimal allocation of resources; see Lange (1942), Samuelson (1947) and Khan (1999) for more details, references, and discussions.

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In the beginning of the 1950s, Arrow (1951) and Debreu (1951) made the next crucial step in the theory of welfare economics considering economic models with possibly nonsmooth but *convex* data. Based on the classical *separation theorems* for convex sets, they and their followers developed a nice theory that, in particular, contains *necessary and sufficient* conditions for Pareto optimal allocations and shows that each of such allocations leads to an *equilibrium* in convex economics. A key result of this theory is the so-called *second fundamental theorem of welfare economics* stated that any Pareto optimal allocation can be associated with a nonzero price vector at which each consumer minimizes his/her expenditure and each firm maximizes its profit; see Debreu (1959). The full statement of this result is due to convexity, which is crucial in the Arrow–Debreu model. Note that the Arrow–Debreu economic theory and related mathematical results have played a fundamental role in developing the general theory of *convex analysis* that is mainly based on convex separation.

However, the relevance of convexity assumptions is often doubtful for many important applications. In particular, these assumptions do not hold in the presence of increasing returns to scale in the production sector, which is widely recognized in the economic literature. In the pioneering study of Guesnerie (1975), a generalized version of the second welfare theorem was established in the form of first-order necessary conditions for Pareto optimal allocations in nonconvex economies. Instead of postulating convexity of the initial production and preference sets, Guesnerie assumed the convexity of their local *tangent* approximations and then employed the classical separation theorem for convex cones. He formalized this procedure by using the 'cone of interior displacements' developed by Dubovitskii and Milyutin (1965) in the general optimization theory.

Guesnerie's approach to the study of Pareto optimality in nonconvex economies was extended in many publications, for both finite-dimensional and infinite-dimensional commodity spaces; see, e.g., Bonnisseau and Cornet (1988), Khan and Vohra (1988), and their references. Most of these publications employ the tangent cone of Clarke (1983) that has an advantage of being *automatically convex* and hence can be treated by using the classical convex separation. In this way, marginal prices are formalized in terms of the dual Clarke normal cone which, however, may be too big for satisfactory results in nonconvex models as clearly demonstrated in Khan (1999).

In the latter paper (its first variant appeared as a preprint of 1987), Khan obtained a more adequate version of the generalized second welfare theorem for nonconvex economies with finite-dimensional commodity spaces. In his version, marginal prices are formalized through the non-convex normal cone of Mordukhovich (1976) that is always contained in Clarke's normal cone and may be significantly smaller in typical nonconvex settings. Note that Khan's approach does not involve any convex separation but employs instead a reduction to necessary optimality conditions in nonsmooth programming obtained in Mordukhovich (1980). In Cornet (1990), similar results were derived for somewhat different economic models by using a direct proof of necessary optimality conditions for the corresponding maximization problems.

In this paper we develop an approach to the study of Pareto optimality in nonconvex economic models that can be viewed as a unification of both the previous approaches discussed above, which are based, respectively, on the reduction to mathematical programming and on the usage of convex separation theorems. The approach of this paper relies on the so-called *extremal principle* in variational analysis that provides necessary conditions for locally extremal points of systems of closed sets and reduces to the classical separation in the case of convexity. Thus, the extremal principle can be treated as a variational extension of the convex separation to the general nonconvex setting. It goes back to the beginning of dual-spaced methods in nonsmooth variational analysis and plays a fundamental role in many aspects of optimization, optimal control, nonconvex calculus, ets.; see the book of Mordukhovich (1988) and the recent study in Mordukhovich (2000a) for more details, discussions, and references.

Based on the extremal principle, we obtain new versions of the generalized second welfare theorem in nonconvex economies with infinite-dimensional commodity spaces. First we establish an *approximate* form of necessary optimality conditions for Pareto and weak Pareto optimal allocations that ensure the existence of approximate marginal prices under general net demand constraint qualifications developing the qualification conditions of Cornet (1986) and Jofré and Rivera (2000). In this result, marginal prices are formalized in terms of Fréchet-like normals at ε-optimal allocations. Then imposing mild sequential normal compactness requirements, we pass to the limit in the approximate conditions and derive an *exact* form of the generalized second welfare theorem in terms of our basic limiting normal cone in Asplund spaces that provides an adequate description of common marginal prices for all the preference, production, and net demand constraint sets at Pareto optimal allocations. In the case of ordered commodity spaces, we justify natural conditions for the marginal price *positivity*. The results obtained bring some new information even in the case of convex economies, since we do not impose either the classical interiority condition or the properness condition of Mas-Colell (1985) (we actually do not need a lattice structure of ordered commodity spaces). On the other hand, our methods and results are not suitable for linear topological spaces that are well covered by the conventional approaches in the convex economic theory.

Note that the usage of Fréchet-like normals in the framework of Asplund spaces allow us to obtain sharper marginal prices and other consequences in comparison with similar results in terms of abstract normals and subgradients developed in Jofré (2000) and Mordukhovich (2000b), where nonconvex separation properties are employed in general Banach spaces. The reader can find more details and discussions in Section 4, where the results obtained compare with classical and recent achievements in this area.

The rest of the paper is organized as follows. In Section 2 we formulate a general model of welfare economics and discuss the basic qualification conditions needed in what follows. Section 3 is devoted to the tools of variational analysis including the extremal principle. Section 4 contains the main results of the paper on necessary conditions for Pareto optimal allocations in nonconvex economies.

Throughout the paper we use standard notation. Let us mention that B and B^* stand, respectively, for the unit closed balls of the Banach space in question and its dual, cl^* signifies the weak-star topological closure in dual spaces, and

$$\begin{aligned}\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \{&x^* \in X^* | \exists \text{ sequences } x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^* \\ &\text{with } x_k^* \in F(x_k), \quad k \in \{\mathbb{N}\}\end{aligned}$$

denotes the *sequential* Painlevé-Kuratowski upper limit for multifunctions $F : X \rightrightarrows X^*$ with respect to the norm topology in X and the weak-star topology in X^*.

2. The Basic Economic Model and Qualification Conditions

In this section we describe a general nonconvex model $\mathcal{E}$ of welfare economics considered in this paper and discuss net demand qualification conditions important in the subsequent study of Pareto optimal allocations.

Let E be a normed *commodity space* of the economy $\mathcal{E}$ that involves n consumers with *consumption sets* $C_i \subset E$, $i = 1, \ldots, n$, and m firms with *production sets* $S_j \subset E$, $j = 1, \ldots, m$. Each consumer has a *preference set* $P_i(x)$ that consists of elements in C_i preferred to x_i by this consumer at the consumption plan $x = (x_1, \ldots, x_n) \in C_1 \times \cdots \times C_n$. So the preference relation in $\mathcal{E}$ is given by n general set-valued mappings $P_i : c - 1 \times \cdots \times C_n \rightrightarrows C_i$ without preordering, utility functions, and conventional assumptions of the classical welfare economics; see, e.g., Debreu (1959). By definition we have $x_i \notin P_i(x)$ for each $i = 1, \ldots, n$, and our underlying assumptions is that at least one consumer is *nonsatiated,* i.e., $P_i(x) \neq \emptyset$.

Let $W \subset E$ be a given nonempty subset of the commodity space called the *net demand constraint set*. This set defines market constraints on feasible allocations of the economy $\mathcal{E}$.

DEFINITION 2.1. Let $x = (x_i) = (x_1, \ldots, x_n)$ and $y = (y_j) = (y_1, \ldots, y_m)$. We say that $(x, y) \in \prod_{i=1}^n C_i \times \prod_{j=1}^m S_j$ is a *feasible allocation* of $\mathcal{E}$ if

$$\sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j \in W. \tag{2.1}$$

Note that W can be formally treated as an additional production set. However, introducing the net constraint set allows us to unify some conventional situations in economic models and to give a useful economic insight in the general framework.

Indeed, in the classical case the set W consists of one element $\{\omega\}$, where ω is an *aggregate endowment* of scarce resources. Then constraint (2.1) reduces to the 'markets clear' condition. Another conventional framework appears in (2.1) when the commodity space E is ordered by a closed positive cone E_+ and $W := \omega - E_+$, which corresponds to the 'implicit free disposal' of commodities. Generally (2.1) describes a natural situation that may particularly happen when the initial aggregate endowment is not exactly known due to, e.g., incomplete information. In the latter general case, the set W reflects some *uncertainty* in the economic model under consideration.

In this paper we consider in parallel the following two notions of Pareto optimal allocations for the economic model $\mathcal{E}$ with the general market constraints (2.1). These abstract notions of Pareto optimality from Definition 2.2 to locally extremal points of some system of closed sets and then to apply the *extremal principle* of variational analysis that can be viewed as an extension of the classical convex separation results to the case of nonconvex sets; see Section 3. For the economic model under consideration, such a reduction becomes possible under certain *qualification conditions* imposed on preference, production, and net demand constraint sets. The following qualification conditions, required respectively for weak Pareto and Pareto optimal allocations in our economic model with general net demand constraints, are in the line of the 'desirability direction condition' of Mas-Colell (1986) and the 'condition (Δ)' of Cornet (1986) used also in Khan (1999) under the name of 'Cornet's constraint qualification'.

DEFINITION 2.2. Let $(\bar{x}, \bar{y})$ be a feasible allocation of the economy $\mathcal{E}$ with the property $\bar{x}_i \in \mathrm{cl} P_i(\bar{x})$ for all $i = 1, \ldots, n$. We say that:

(i) $(\bar{x}, \bar{y})$ is a *weak Pareto optimal allocation* of $\mathcal{E}$ if there is no feasible allocation (x, y) such that $x_i \in P_i(\bar{x})$ for all $i = 1, \ldots, n$.

(ii) $(\bar{x}, \bar{y})$ is a *Pareto optimal allocation* of $\mathcal{E}$ if there is no feasible allocation (x, y) such that $x_i \in \mathrm{cl} P_i * \bar{x})$ for all $i = 1, \ldots, n$ and $x_i \in P_i(\bar{x})$ for at least one i.

Similarly we define Pareto and weak Pareto *local* optimal allocations when feasible allocations in Definition 2.2 are restricted by some neighborhood O of $(\bar{x}, \bar{y})$. Note that Pareto optimal allocations may be different from their weak counterparts only if $n > 1$.

The principal objective of this paper is to obtain necessary optimality conditions for Pareto and weak Pareto local optimal allocations of the economy $\mathcal{E}$. To furnish this, we are going to reduce the generalized notions of Pareto optimality from Definition 2.2 to locally extremal points of some system of closed sets and then to apply the *extremal principle* of variational analysis that can be viewed as an extension of the classical convex separation results to the case of nonconvex sets; see Section 3. For the economic model under consideration, such a reduction becomes possible under certain *qualification conditions* imposed on preference, production, and net demand constraint sets. The following qualification conditions, required respectively for weak Pareto and Pareto optimal allocations in our economic model with general net demand constraints, are in the line of the ‘desirability direction condition’ of Mas-Colell (1986) and the ‘condition (Δ)’ of Cornet (1986) used also in Khan (1999) under the name of ‘Cornet’s constraint qualification’.

DEFINITION 2.3. Let $(\bar{x}, \bar{y})$ be a feasible allocation of $\mathcal{E}$ and let

$$\bar{\omega} := \sum_{i=1}^{n} \bar{x}_i - \sum_{j=1}^{m} \bar{y}_j. \tag{2.2}$$

Given $\varepsilon > 0$, we consider the set

$$\Delta_\varepsilon := \sum_{i=1}^{n} \mathrm{cl} P_i(\bar{x}) \cap (\bar{x}_i + \varepsilon B) - \sum_{j=1}^{m} \mathrm{cl} S_j \cap (\bar{y}_j + \varepsilon B) - \mathrm{cl} W \cap (\bar{w} + \varepsilon B) \tag{2.3}$$

and say that:

(i) The *net demand weak qualification (NDWQ) condition* holds at $(\bar{x}, \bar{y})$ if there are $\varepsilon > 0$ and a sequence $\{e_k\} \subset E$ with $e_k \to 0$ as $k \to \infty$ such that

$$\Delta_\varepsilon + e_k \subset \sum_{i=1}^{n} P_i(\bar{x}) - \sum_{j=1}^{m} S_j - W \tag{2.4}$$

for all $k \in \mathbb{N}$ sufficiently large.

(ii) The *net demand qualification (NDQ) condition* holds at $(\bar{x}, \bar{y})$ if there are $\varepsilon > 0$, a sequence $\{e_k\} \subset X$ with $e_k \to 0$ as $k \to \infty$, and a consumer index $i_0 \in \{1, \ldots, n\}$ such that

$$\Delta_\varepsilon + e_k \subset P_{i_0}(\bar{x}) + \sum_{i \neq i_0} \mathrm{cl} P_i(\bar{x}) - \sum_{j=1}^{m} S_j - W \tag{2.5}$$

for all $k \in \mathbb{N}$ sufficiently large.

Obviously the NDWQ condition implies the NDQ one, but the opposite is not true. When $W = \{\omega\}$ (the markets clear) and all the production sets S_j are locally closed, the NDQ condition reduces to the ‘asymptotically included condition’ of Jofré and Rivera (2000), which directly implies (2.5) in the general case under consideration. So the sufficient conditions for the latter property presented in Jofré (2000) and Jofré and Rivera (2000) as well as those for Cornet’s constraint qualification presented in Cornet (1986) and Khan (1999) in finite dimensions, ensure the validity of the net demand qualification condition (2.5). Note that Cornet’s constraint qualification corresponds to (2.5) with no set W, where e_k is replaced with te for some $e \in E$ and all $t > 0$ sufficiently small. The latter property holds, in particular, if either one among preference or production sets is epi-Lipschitzian at the corresponding point in the sense of Rockafellar (1980).

Recall that a subset $\Omega \subset X$ of a normed space X is *epi-Lipschitzian* at $\bar{x} \in \mathrm{cl}\Omega$ if there are a vector $v \in X$ and a number $\gamma > 0$ such that

$$x + t(v + \gamma B) \subset \Omega \text{ for all } x \in (\bar{x} + \gamma B) \cap \Omega \text{ and } t \in (0, \gamma). \tag{2.6}$$

When $v = 0$, property (2.6) is obviously equivalent to $\bar{x} \in \mathrm{int}\Omega$. If $v \neq 0$ and Ω is closed, the epi-Lipschitzian property means that Ω is locally homeomorphic to the epigraph of a Lipschitz continuous function; hence the terminology. Note that the epi-Lipschitzian property of Ω at $\bar{x}$ implies this property of the *closure* $\mathrm{cl}\Omega$ at the same point, but not vice versa. It is worth mentioning that *summation* of sets as in (2.8) and (2.9) below (especially for a large number of sets) tends to improve properties related to nonempty interior, and that the epi-Lipschitzian property of sets falls into this category. If Ω is closed and convex, this property reduces to $\mathrm{int}\Omega \neq \emptyset$. For general closed sets in finite dimensions, the epi-Lipschitzian property is equivalent to the nonempty interior of the Clarke tangent cone to Ω at $\bar{x}$; see Rockafellar (1980) for more details.

The next proposition presents some sufficient conditions for the NDWQ and NDQ properties and particularly demonstrates the difference between the assumptions needed for these properties.

PROPOSITION 2.4. *Let E be a normed space and let $(\bar{x}, \bar{y})$ be a feasible allocation of the economy $\mathcal{E}$. The following assertions hold:*

(i) *Assume that the sets S_j, $j = 1, \dots, m$, and W are closed near the points $\bar{y}_j$ and $\bar{w}$ respectively. Then the NDQ condition is satisfied at $(\bar{x}, \bar{y})$ if there are a number $\varepsilon > 0$, an index $i \in \{1, \dots, n\}$ and a desirability sequence $\{e_{ik}\} \subset E$, $e_{ik} \to 0$ as $k \to \infty$, such that*

$$cl P_i(\bar{x}) \cap (\bar{x}_i + \varepsilon B) + e_{ik} \subset P_i(\bar{x}) \text{ for all large } k \in \mathbb{N}. \tag{2.7}$$

Moreover, the NDQ condition is satisfied at $(\bar{x}, \bar{y})$ if a desirability sequence $\{v_{ik}\}$ exists for each $i \in \{1, \dots, n\}$ with some $\varepsilon > 0$ in (2.7).

(ii) *Assume that $\bar{x}_i \mathrm{cl} P_i(\bar{x})$ for all $i = 1, \dots, n$. Then the NDWQ condition is satisfied at $(\bar{x}, \bar{y})$ if the set*

$$\Delta := \sum_{i=1}^{n} P_i(\bar{x}) - \sum_{j=1}^{m} S_j - W \tag{2.8}$$

is epi-Lipschitzian at $0 \in \mathrm{cl}\Delta$. It happens when either one among the sets $P_i(\bar{x})$ for $i = 1, \dots, n$, S_j for $j = 1, \dots, m$, and W or some of their partial combinations in (2.8) is epi-Lipschitzian at the corresponding point.

(iii) *Assume that $n > 1$. The NDQ condition is satisfied at $(\bar{x}, \bar{y})$ if there is a nonsatiated consumer $i_0 \in \{1, \dots, n\}$ such that the set*

$$\Sigma := \sum_{i \neq i_0} \mathrm{cl} P_i(\bar{x}) \tag{2.9}$$

is epi-Lipschitzian at the point $\sum_{i\neq i_0}\bar{x}_i$. *It happens when either one among the sets* $\mathrm{cl}P_i(\bar{x})$ *for* $i\in\{1,\ldots,n\}\setminus\{i_0\}$ *or some of their partial sums is epi-Lipschitzian at the corresponding point.*

Proof. Both statements in (i) follow immediately from the definitions and the assumptions made. Note that (2.7) is a direct generalization of the desirability direction condition in Mas-Colell (1986); it is related to the classical 'more is better' assumption for convex economies with commodity spaces ordered by their closed positive cones having nonempty interiors.

Let us prove (ii). Due to the structure of (2.4), it is sufficient to consider the case when the aggregate set Δ in (2.8) is epi-Lipschitzian at the origin. Using (2.6) in the Banach space E, we find $v\in E$ and $\gamma>0$ such that

$$\Delta\cap(\gamma B)+t(v+\gamma B)\subset\Delta \text{ for all } t\in(0,\gamma). \tag{2.10}$$

Picking an arbitrary sequence $t_k\downarrow 0$ as $k\to\infty$, we put

$$e_k:=t_k v,\ k\in\mathbb{N},\quad \text{and}\quad \varepsilon:=\gamma/(n+m+2) \tag{2.11}$$

and show that the NDWQ condition (2.4) holds with e_k and ε in (2.11). To furnish this, we take any $z_\varepsilon\in\Delta_\varepsilon$ and conclude, by (2.3) and (2.2), that $z_\varepsilon\in(n+m+1)\varepsilon B$. Due to the structure of the sets Δ_ε in (2.3) and Δ in (2.8), we find a sequence of elements $z_k\in\Delta$ converging to z_ε as $k\to\infty$. Obviously

$$z_k\in(n+m+2)\varepsilon B=\gamma B \text{ for large } k\in\mathbb{N} \tag{2.12}$$

due to the choice of ε in (2.11). We can also select z_k so that

$$z_\varepsilon-z_k\in(t_k\gamma)B \text{ for large } k\in\mathbb{N}. \tag{2.13}$$

Now combining (2.10)–(2.13), we get

$$\begin{aligned} z_\varepsilon+e_k &= z_k+t_k v+(z_\varepsilon-z_k)\\ &\in\Delta\cap(\gamma B)+t_k(v+\gamma B)\subset\Delta, \end{aligned}$$

which implies (2.4).

It remains to prove (iii) considering the case when the set Σ in (2.9) is epi-Lipschitzian at the reference point. Using this property, we find $v\in E$ and $\gamma>0$ such that

$$\sum_{i\neq i_0}\mathrm{cl}P_i(\bar{x})\cap\left(\sum_{i\neq i_0}\bar{x}_i+\gamma B\right)+t(v+\gamma B)\subset\sum_{i\neq i_0}\mathrm{cl}P_i(\bar{x}). \tag{2.14}$$

Now select v_k and ε as in (2.11) and proceed similarly to the above proof of (ii). Take $z_\varepsilon\in\Delta_\varepsilon$ with

$$z_\varepsilon=\sum_{i=1}^{n}x_i-\sum_{j=1}^{m}y_j-w,\quad x_i\in\mathrm{cl}P_i(\bar{x}),\ y_j\in\mathrm{cl}S_j,\ w\in\mathrm{cl}W,$$

and approximate x_{i_0}, y_j, and w by sequences of elements from the corresponding sets $P_{i_0}(\bar{x})$, S_j, and W. In contrast to the proof of (ii), we do not approximate x_i for $i \neq i_0$. In this way we derive the net demand qualification condition (2.5) from the epi-Lipschitzian property (2.14) and complete the proof of (iii). □

It is important to observe that we do *not need* to impose any assumption on preference and production sets for the fulfillment of both qualification conditions in Definition 2.3 if the net demand constraint set W is epi-Lipschitzian at the point $\bar{w}$ in (2.2). This follows from Proposition 2.4(ii). It happens, in particular, when E is ordered and $W = \omega - E_+$ with $\text{int}E_+ \neq \emptyset$, which is the classical case of 'free-disposal Pareto optimum.' Note also that the material of this section holds, with minor modifications, in general *linear topological spaces* equipped with a locally convex Hausdorff topology.

3. Normal Cones and the Extremal Principle

In this section we present the basic tools of variational analysis used in the paper for studying Pareto optimal allocations of the nonconvex economy $\mathcal{E}$ from Section 2. Our main results are obtained in Section 4 in terms of generalized normals to nonconvex sets. Let us start with the description of the basic normal constructions in Banach spaces used in the sequel.

DEFINITION 3.1. Let Ω be a nonempty subset of a Banach space X and let $\varepsilon \geqslant 0$.

(i) Given $x \in \Omega$, we define the *set of ε-normals* to Ω at x by

$$\hat{N}_\varepsilon(x; \Omega) := \left\{ x^* \in X^* \,\middle|\, \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leqslant \varepsilon \right\}, \tag{3.1}$$

where $u \to^{\Omega} x$ means that $u \to x$ with $u \in \Omega$. When $\varepsilon = 0$, the set (3.1) is a cone called the *prenormal cone* or the *Fréchet normal cone* to Ω at x and denoted by $\hat{N}(x; \Omega)$. If $x \notin \Omega$, we put $\hat{N}_\varepsilon(x; \Omega) := \emptyset$ for all $\varepsilon \geqslant 0$.

(ii) The conic set

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{x \to \bar{x}, \varepsilon \downarrow 0} \hat{N}_\varepsilon(x; \Omega) \tag{3.2}$$

is called the (basic) *normal cone* to Ω at $\bar{x}$.

In the finite-dimensional case $X = \mathbb{R}^n$, the basic normal cone (3.2) coincides with the one introduced in Mordukhovich (1976) by

$$N(\bar{x}; \Omega) = \operatorname*{Lim\,sup}_{x \to \bar{x}} [\text{cone}(x - \Pi(x; \Omega))], \tag{3.3}$$

where 'cone' stands for the conic hull of a set and $\Pi(x; \Omega)$ is the multivalued Euclidean projector of x on the closure of Ω.

The set of ε-normals (3.1) and the extension (3.2) of the basic normal cone to Banach spaces first appeared in Kruger and Mordukhovich (1980). Observe that although the set (3.1) is always nonconvex for every $\varepsilon \geqslant 0$, the basic normal cone may be *nonconvex* in common situations, e.g., for $\Omega = \text{gph}|x|$ at $\bar{x} = (0, 0) \in \mathbb{R}^2$, This means that it cannot be dual to any tangent cone approximation of Ω at the point in question.

Let us mention that for $X = \mathbb{R}^n$ the dual/polar cone to $\hat{N}(\bar{x}; \Omega)$ coincides to the classical Bouligand-Severi contingent cone, while the convex closure of $N(\bar{x}; \Omega)$ agrees with the normal cone of Clarke (1983). Despite its nonconvexity, the basic normal cone in finite dimensions enjoys a number of nice properties some of which may be spoiled by the convexification procedure; see the books of Mordukhovich (1988) and Rockafellar and Wets (1998) for more details, discussions, and references.

In the case of infinite dimensions, most of these properties hold true under natural assumptions for a broad subclass of Banach spaces, called *Asplund spaces*, on which every continuous convex functions is generically Fréchet differentiable. By now this class is well investigated in the geometric theory of Banach spaces, where many useful properties and characterizations of Asplund spaces have been obtained; see, e.g., the book of Phelps (1993) and its references. In particular, Asplund spaces are characterized as those for which every separable subspace has a separable dual, and they include Banach spaces with Fréchet differentiable renorms or bump functions (hence, all reflexive spaces). On the other hand, there are Asplund spaces that fail to have even a Gâteaux differentiable renorm.

If $\Omega \subset X$ is a *convex* subset of a Banach space X, then both prenormal and normal cones in Definition 3.1 reduce to the normal cone of convex analysis:

$$N(\bar{x}; \Omega) = \hat{N}(\bar{x}; \Omega) = \{x^* \in X^* | \langle x^*, x - \bar{x}\rangle \leqslant 0 \quad \forall x \in \Omega\} \tag{3.4}$$

Moreover, for convex sets one has

$$\hat{N}_\varepsilon(\bar{x}; \Omega) = \hat{N}(\bar{x}; \Omega) + \varepsilon B^* = \{x^* \in X^* | \langle x^*, x - \bar{x}\rangle \leqslant \varepsilon\|x - \bar{x}\| \quad \forall x \in \Omega\}. \tag{3.5}$$

If Ω is an arbitrary closed subset of an Asplund space X, then there is the exact relationship between the prenormal and normal cones proved in Mordukhovich and Shao (1996b):

$$N(\bar{x}; \Omega) = \mathop{\text{Lim}\,\text{sup}}_{x \to \bar{x}} \hat{N}(x; \Omega). \tag{3.6}$$

It is also shown in the latter paper that the weak-star *topological closure* $\text{cl}^* N(\bar{x}; \Omega)$ of (3.2) gives the 'approximate' G-normal cone of Ioffe (1989) while the weak-star closure of its *convexification* $\text{cl}^*\text{co} N(\bar{x}; \Omega)$ coincides with Clarke's normal cone for any closed sets in Asplund spaces. Note that our basic sequential normal cone (3.2) may be strictly smaller than the G-normal cone (and its 'nucleus' called the ‘approximate normal cone’ in Ioffe (2000)) even in spaces with Fréchet smooth norms; see examples in Borwein and Fitzpatrick (1995).

In the sequel we use the following two propositions involving the prenormal cone $\hat{N}(\cdot;\Omega)$. The first one can be easily derived from the definition.

PROPOSITION 3.2. *Let X_1 and X_2 be Banach spaces. Then for any nonempty sets $\Omega_i \subset X_i$, $i = 1, 2$, and any $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \Omega_1 \times \Omega_2$ one has*

$$\hat{N}(\bar{x}; \Omega_1 \times \Omega_2) = \hat{N}(\bar{x}; \Omega_1) \times \hat{N}(\bar{x}_2; \Omega_2).$$

The next proposition provides variational descriptions of the prenormal cone and the set of ε-normals (3.1) useful for economic interpretations of some results obtained in Section 4. Given a class $\mathcal{S}$ of smooth in a certain sense functions on the Banach space X, we recall that $b : X \to \mathbb{R}$ is an *$\mathcal{S}$-hump function* on X if $b(\cdot) \in \mathcal{S}$, $b(x_0) \neq 0$ for some $x_0 \in X$, and $b(x) = 0$ whenever x lies outside a ball in X. In what follows, we consider the three classes $\mathcal{S}$ of smooth functions on X: Fréchet smooth functions, Lipschitzian and Fréchet smooth functions, and Lipschitzian and continuously differentiable functions. Note that every Banach space X admitting an $\mathcal{S}$-bump function from one of these three classes must be Asplund.

PROPOSITION 3.3. *Let Ω be a subset of a Banach spaces X and let $\bar{x} \in \Omega$. Then the following hold:*

(i) *Given $\varepsilon \geqslant 0$, we have $x^* \in \hat{N}_\varepsilon(\bar{x};\Omega)$ if and only if for any $\gamma > 0$ the function*

$$f(x) := \langle x^*, x - \bar{x}\rangle - (\varepsilon + \gamma)\|x - \bar{x}\|$$

attains at $\bar{x}$ a local maximum relative to Ω.

(ii) *Let X admit an equivalent Fréchet smooth norm. Then for every $x^* \in \hat{N}(\bar{x};\Omega)$ there is a concave Fréchet smooth function $g : X \to \mathbb{R}$ such that $\nabla g(\bar{x}) = x^*$ and $g(\cdot)$ achieves its global maximum relative to Ω uniquely at $\bar{x}$.*

(iii) *Let X admit an $\mathcal{S}$-smooth bump function. Then for every $x^* \in \hat{N}(\bar{x};\Omega)$ there is an $\mathcal{S}$-smooth function $g : X \to \mathbb{R}$ satisfying the conclusions in (ii).*

Proof. The proof of (i) follows directly from the definition of ‘lim sup’ in (3.1). Assertions (ii) and (iii) follow from the proof of Theorem 4.6 in Fabian and Mordukhovich (1998) for the case of set indicator functions; see also Remarks 4.9 and 4.10 therein. □

Our basic tool in this paper in the following *extremal principle* of variational analysis that provides necessary optimality conditions for locally extremal points of set systems and can be viewed as a variational extension of the classical separation principle to the case of nonconvex sets. The reader may consult with the survey in Mordukhovich (2000a) for more references and discussions.

DEFINITION 3.4. Let $\Omega_1, \ldots, \Omega_n (n \geqslant 2)$ be nonempty subsets of a Banach space X. We say that $\bar{x}$ is a *locally extremal point* of the set system $\{\Omega_1, \ldots, \Omega_n\}$

if there are sequences $\{a_{ik}\} \subset X$, $i = 1, \dots, n$, and a neighborhood U of $\bar{x}$ such that $a_{ik} \to 0$ as $k \to \infty$ and

$$\bigcap_{i=1}^{n} (\Omega_i - a_{ik}) \cap U = \emptyset \text{ for all large } k \in \mathbb{N}.$$

We say that $\{\Omega_1, \dots, \Omega_n\}$ is an *extremal system* in X if these sets have at least one locally extremal point.

An obvious example of the extremal system of two sets is provided by the pair $\{\bar{x}, \Omega\}$, where $\bar{x}$ is a boundary point of the closed set $\Omega \subset X$. In general, this geometric concept of extremality covers conventional notions of optimal solutions to various problems of scalar and vector optimization.

The following fuzzy version of the extremal principle was established Mordukhovich and Shao (1996a) as a characterization of Asplund spaces.

THEOREM 3.5. *Let X be an Asplund space and let $\bar{x}$ be a locally extremal point of the system of closed sets $\{\Omega_1, \dots, \Omega_n\}$ in X. Then for any $\varepsilon > 0$ there are $x_i \in \Omega_i \cap (\bar{x} + \varepsilon B)$ and $x_i^* \in X^*$ such that*

$$x_i^* \in \hat{N}(x_i; \Omega_i) + \varepsilon B^* \text{ for all } i = 1, \dots, n, \tag{3.7}$$

$$x_1^* + \cdots + x_n^* = 0, \tag{3.8}$$

$$\|x_1^*\| + \cdots + \|x_n^*\| = 1. \tag{3.9}$$

Let us discuss conditions under which one can pass to the limit in (3.7)–(3.9) as $\varepsilon \downarrow 0$ and obtain the exact/limiting form of the extremal principle in terms of the basic normal cone (3.6) in Asplund spaces. To furnish the limiting process, we need additional compactness assumptions involving Fréchet normals. The following general condition was formulated in Mordukhovich and Shao (1996c) although it has been actually used earlier for similar procedures.

DEFINITION 3.6. Let $\Omega \subset X$ be a nonempty subset of the Banach space X and let $\bar{x} \in \Omega$. The set Ω is said to be *sequentially normally compact* (SNC) at $\bar{x}$ if for any sequence (x_k, x_k^*) satisfying

$$x_k^* \in \hat{N}(x_k; \Omega), \quad x_k \to \bar{x}, \quad \text{and} \quad x_k^* \xrightarrow{w^*} 0$$

one has $\|x_k^*\| \to 0$ as $k \to \infty$.

Sufficient conditions for SNC closed sets are provided by the ‘compactly epi-Lipschitz’ (CEL) property introduced in Borwein and Strojwas (1985) as an extension of the epi-Lipschitzian behavior (2.6). Recently efficient characterizations of the CEL property were obtained in Borwein, Lucet and Mordukhovich (2000) for closed convex sets in normed spaces. One of these characterizations says that Ω is CEL if and only if its span is a finite-codimensional closed subspace and the relative interior of Ω (with respect to its affine hull) is nonempty. In Ioffe (2000), other characterizations of the CEL property are obtained for general closed sets in terms of normal cones satisfying certain requirements in appropriate Banach spaces. Note that both SNC and CEL properties automatically hold in finite dimensions.

Using the SNC property and following the proof of Theorem 3.6 in Mordukhovich and Shao (1996b), we arrive at the limiting version of the extremal principle in terms of our basic normal cone.

COROLLARY 3.7. *Let $\bar{x}$ be a locally extremal point of the closed set system $\{\Omega_1, \dots, \Omega_n\}$ in the Asplund space X. Assume that all but one of the sets $\Omega_1, \dots, \Omega_n$ are sequentially normally compact at $\bar{x}$. Then there are $(x_1^*, \dots, x_n^*) \neq 0$ satisfying (3.8) and*

$$X_i^* \in N(\bar{x}; \Omega_i) \text{ for all } i = 1, \dots, n. \tag{3.10}$$

Due to the normal cone representation (3.4) in the case of *convex* sets, relations (3.8)–(3.10) of the extremal principle for $n = 2$ reduce to the classical *separation property*

$$\exists x^* \neq 0 \text{ with } \langle x^*, x_1\rangle \leqslant \langle x^*, x_2\rangle \text{ for all } x_1 \in \Omega_1 \text{ and } x_2 \in \Omega_2. \tag{3.11}$$

This means that the extremal principle ensures the separation property for two convex sets imposing the SNC assumption on one of them instead of the more restrictive nonempty interior assumption in the classical separation theorem. Note that Corollary 3.7 provides the above relations only for *locally extremal points*. However, the latter requirement holds *automatically* (or under mild assumptions) in applications to various optimization problems, calculus rules in nonsmooth analysis, economic models, etc.; see more discussions in Mordukhovich (2000a). On the other hand, we can easily check that the separation property (3.11) implies the local extremality of any point $\bar{x} \in \Omega_1 \cap \Omega_2$ for arbitrary closed sets. Thus, the result of Corollary 3.7 turns out to be a proper *variaitonal extension* of the convex separation theorem to a broad nonconvex setting and, moreover, provides an improvement of the classical results in the case of convex sets.

We refer the reader to Mordukhovich (2000b) for more general analogs of the extremal principle in fuzzy/approximate and exact/limiting forms obtained in terms of abstract prenormal and normal structures under minimal assumptions in appropriate Banach spaces.

4. Generalized Second Welfare Theorems

This section contains the main results of the paper on necessary conditions for Pareto optimal allocations of the nonconvex economy $\mathcal{E}$. Based on the extremal principle, we obtain necessary conditions for Pareto and weak Pareto optima in terms of the prenormal and normal cones of Definition 3.1. These conditions are presented in both approximate and exact forms of the *generalized second welfare theorem* involving *common* nonzero marginal prices for all the preference and production sets. We discuss various corollaries and interpretations of the main results and derive new conditions of the *price positivity* for economies with ordered commodity spaces.

First let us establish a generalized version of the second welfare theorem in the *approximate* form for Pareto and weak Pareto optimal allocations o f $\mathcal{E}$ under the corresponding net demand qualification conditions of Definition 2.3.

THEOREM 4.1. *Let* $(\bar{x}, \bar{y})$ *be a Pareto (weak Pareto) local optima allocation of the economy* $\mathcal{E}$ *with the Asplund commodity space* E. *Assume that the net demand qualification condition (resp. net demand weak qualification condition) is satisfied at* $(\bar{x}, \bar{y})$. *Then for every* $\varepsilon > 0$ *there are* $(x, y, w) \in \prod_{i=1}^{n} \mathrm{cl} P_i(\bar{x}) \times \prod_{j=1}^{m} \mathrm{cl} S_j \times \mathrm{cl} W$ *and* $p^* \in E^* \setminus \{0\}$ *such that*

$$-p^* \in \hat{N}(x_i; \mathrm{cl} P_i(\bar{x})) + \varepsilon B^* \text{ with } x_i \in \bar{x}_i + (\varepsilon/2)B \text{ for all } i = 1, \ldots, n, \tag{4.1}$$

$$p^* \in \hat{N}(y_j; \mathrm{cl} S_j) + \varepsilon B^* \text{ with } y_j \in \bar{y}_j + (\varepsilon/2)B \text{ for all } j = 1, \ldots, m, \tag{4.2}$$

$$p^* \in \hat{N}(w; \mathrm{cl} W) + \varepsilon B^* \text{ with } w \in \bar{w} + (\varepsilon/2)B, \tag{4.3}$$

$$\frac{1-\varepsilon}{2\sqrt{n+m+1}} \leqslant \|p^*\| \leqslant \frac{1+\varepsilon}{2\sqrt{n+m+1}}, \tag{4.4}$$

where $\bar{w}$ *is defined in (2.2).*

Proof. We prove the theorem in a parallel way for Pareto and weak Pareto optimal allocations $(\bar{x}, \bar{y})$. Consider the product space $X := E^{n+m+1}$ equipped with the norm

$$\|(v_1, \ldots, v_{n+m+1})\|_X := [\|v_1\|^2 + \cdots + \|v_{n+m+1}\|^2]^{1/2}. \tag{4.5}$$

Since E is Asplund, the product space X is Asplund as well; see, e.g., Phelps (1993). Taking a number $\varepsilon > 0$ for which the NDQ condition (resp. the NDWQ condition) holds with the corresponding sequence $\{e_k\}$ in (2.5) and (2.4), we define the two closed sets in X as follows:

$$\Omega_1 := \prod_{i=1}^{n} [\mathrm{cl} P_i(\bar{x}) \cap (\bar{x}_i + \varepsilon B)] \times \prod_{j=1}^{m} [\mathrm{cl} S_j \cap (\bar{y}_j + \varepsilon B)] \times [\mathrm{cl} W \cap (\bar{w} + \varepsilon B)], \tag{4.6}$$

$$\Omega_2 := \left\{ (x, y, w) \in X \,\middle|\, \sum_{i=1}^{n} x_i - \sum_{j=1}^{m} y_j - w = 0 \right\}. \tag{4.7}$$

Let us show that $(\bar{x}, \bar{y}, \bar{w})$ is a locally extremal point of the set system $\{\Omega_1, \Omega_2\}$ in (4.6) and (4.7). It follows directly from Definitions 2.1 and 2.2 that $(\bar{x}, \bar{y}, \bar{w}) \in \Omega_1 \cap \Omega_2$. To justify the local extremality of $(\bar{x}, \bar{y}, \bar{w})$ it is sufficient to find a neighborhood U of this point and a sequence $\{a_k\} \subset X$ such that $a_k \to 0$ as $k \to \infty$ and

$$(\Omega_1 - a_k) \cap \Omega_2 \cap U = \emptyset \text{ for all large } k \in \mathbb{N} \tag{4.8}$$

under the corresponding qualification condition from Definition 2.3. To proceed, we take a neighborhood $O \in E^{n+m}$ of the Pareto (weak Pareto) optimal allocation $(\bar{x}, \bar{y})$ and a sequence $\{e_k\} \subset E$ converging to zero for which either (2.4) or (2.5) is satisfied. In both cases we put $U := O \times \mathbb{R} \subset X$ and $a_k := (0, \ldots, 0, e_k) \in X$ and show that (4.8) holds for the same $k \in \mathbb{N}$ as in (2.4) and (2.5). Assuming the contrary, we find $z_k \in \Omega_1$ with $z_k - a_k \in \Omega_2$. Due to the structure of (4.6) and (4.7) and the construction of a_k and U, this implies the existence of (x_k, y_k, w_k) with $(x_k, y_k) \in O$,

$$x_{ik} \in \text{cl}P_i(\bar{x}) \cap (\bar{x}_i + \varepsilon B),\ \ i = 1, \ldots, n;$$
$$y_{jk} \in \text{cl}S_j \cap (\bar{y}_j + \varepsilon B),\ \ j = 1, \ldots, m,$$

$w_k \in \text{cl}W \cap (\bar{w} + \varepsilon B)$, and

$$\sum_{i=1}^{n} x_{ik} - \sum_{j=1}^{m} y_j - k - w_k + e_k = 0.$$

The latter means, due to (2.3), that $0 \in \Delta_\varepsilon + e_k$. Consequently, the NDWQ condition ensures that the right-hand side set in (2.4) contains the origin while the NDQ condition ensures that the origin belongs to the right-hand side set in (2.5). This contradicts the weak Pareto local optimality of $(\bar{x}, \bar{y})$ in the first case and the Pareto local optimality of this allocation in the second case. Thus we have justified (4.8) and established the local extremality of $(\bar{x}, \bar{y}, \bar{w})$ for the system of closed sets $\{\Omega_1, \Omega_2\}$ defined in (4.6) and (4.7) in the Asplund space X.

Now we can apply the fuzzy version of the extremal principle to this system of two sets. According to Theorem 3.5, for every $\varepsilon > 0$ there are $z = (x_1, \ldots, x_n, y_1, \ldots, y_m, w) \in \Omega_1$, $\tilde{z} \in \Omega_2$,

$$z^* \in \hat{N}(z; \Omega_1), \quad \text{and} \quad \tilde{z}^* \in \hat{N}(\tilde{z}; \Omega_2) \tag{4.9}$$

such that

$$\begin{aligned} & x_i \in \bar{x}_i + (\varepsilon/2)B, i = 1, \ldots, n; \\ & y_j \in \bar{y}_j + (\varepsilon/2)B, j = 1, \ldots, m; \\ & w \in \bar{w} + (\varepsilon/2)B, \end{aligned} \tag{4.10}$$

$$(1-\varepsilon)/2 \leqslant \|\tilde{z}^*\| \leqslant (1+\varepsilon)/2, \quad \text{and} \quad \|z^* + \tilde{z}^*\| \leqslant \varepsilon/2. \tag{4.11}$$

Observe that the set Ω_2 in (4.7) is a linear subspace separated in all the variables (x_i, y_j, w). Thus $\hat{N}(\tilde{z}; \Omega_2)$ is a subspace orthogonal to Ω_2, and $\tilde{z}^* = (p^*, \dots, p^*, -p^*, \dots, -p^*)$ in (4.9), where the 'minus terms' start with the $(n+1)$st position. It follows from (4.5) and (4.11) that

$$(1-\varepsilon)/2 \leqslant \sqrt{n+m+1}\|p^*\| \leqslant (1+\varepsilon)/2. \tag{4.12}$$

Then we conclude from (4.9) and the last estimate in (4.111) that

$$-\tilde{z}^* = (-p^*, \dots, -p^*, p^*, \dots, p^*) \in \hat{N}(z; \Omega_1) + \varepsilon B^*. \tag{4.13}$$

Now we use Proposition 3.2 for the product set Ω_1 and observe that by (4.10) all the components (x_i, y_j, w) of the point z in (4.13) belong to the interiors of the corresponding neighborhoods in (4.6); hence these neighborhoods can be ignored in the calculation of $\hat{N}(z; \Omega_1)$. Finally combining (4.10), (4.12), and (4.13), we arrive at relationships (4.1)–(4.4) and complete the proof of the theorem. □

Observe that, in contrast to the fuzzy extremal principle of Theorem 3.5 for the general extremal system of closed sets, Theorem 4.1 ensures the existence of a *common* dual element $p^* \in E^* \setminus \{0\}$ for *all* the sets involved in (4.1)–(4.3), instead of different elements x_i^* in (3.7)–(3.9). This common element, which can be interpreted as an approximate *marginal price* for all the preference and production sets near Pareto optimal allocations, corresponds to the very essence of the classical second welfare theorem ensuring the equality of marginal rates of substitution for consumers and firms. Note that such a specification of the general extremal principle in the economic model under consideration is proved to be possible due to the specific structure of sets (4.6) and (4.7) in the extremal system, especially due to the *separated* variables in (4.7).

the results of Theorem 4.1 can be compared with a recent 'viscous' version of the generalized second welfare theorem established in Jofré (2000) for the case of Pareto optimal allocations in nonconvex economies with the 'market clear' condition $W = \{\omega\}$. The main result of the latter paper is expressed in terms of an abstract subdifferential for Lipschitz continuous functions on a Banach space under certain requirements. Observe that not all of these requirements (particularly the subdifferential sum rule) are satisfied for the Fréchet subdifferential in Asplund spaces that generates the prenormal cone $\hat{N}(\cdot; \Omega)$ through the distance function to Ω, Thus, our Theorem 4.1 and Theorem 3 of Jofré (2000) are independent. The proof of the latter result is based on a subdifferential condition for boundary points of the sum of closed sets from Borwein and Jofré (1998), which is an approximate version of the nonconvex separation property established in Jofré and Rivera (2000) in finite dimensions as an extension of the unpublished result by Cornet and Rockafellar (1989). An abstract version of Theorem 4.1 that covers the mentioned result of Jofré is obtained in Mordukhovich (2000b) by using a more general fuzzy extremal principle.

Let us present some corollaries of Theorem 4.1 taking into account special descriptions of prenormal elements given in Section 3. First we consider economies with *convex* preference and production sets. In this c ase relations (4.1) and (4.2) provide, respectively, *global minimization* (*maximization*) of the *perturbed* consumer expenditures (firm profits) over the corresponding preference (production) sets.

COROLLARY 4.2. *In addition to the assumptions of Theorem 4.1, we suppose that all the sets* $P_i(\bar{x}), i = 1, \dots, n$, *and* S_j, $j = 1, \dots, m$, *are convex. Then for every* $\varepsilon > 0$ *there are* $(x, y, w) \in \prod_{i=1}^{n}[\mathrm{cl}P_i(\bar{x}) \cap (\bar{x}_i + (\varepsilon/2)B)] \times \prod_{j=1}^{m}[\mathrm{cl}S_j \cap (\bar{y}_j + (\varepsilon/2)B] \times \mathrm{cl}W$ *and* $p^* \in E^*\backslash\{0\}$ *such that one has (4.3), (4.4), and*

$$\langle p^*, u_i - x_i\rangle \geqslant -\varepsilon\|u_i - x_i\| \text{ for all } u_i \in \mathrm{cl}P_i(\bar{x}), \quad i = 1, \dots, n, \tag{4.14}$$

$$\langle p^*, v_j - y_j\rangle \leqslant \varepsilon\|v_j - y_j\| \text{for all } v_j \in \mathrm{cl}S_j, \quad j = 1, \dots, m. \tag{4.15}$$

Proof. It follows directly from (4.10, (4.2), and the representation (3.5) of ε-normals to convex sets. □

Next we consider a general case of nonconvex economies and use the variational descriptions of Fréchet normals and ε-normals from Proposition 3.3. In this way we obtain two nonconvex counterparts of Corollary 4.2. The first one provides *local* analogs of relations (4.14) and (4.15) for nonconvex economies, while the second statement ensures the existence of *smooth* functions whose *rate of change* at ε-optimal allocations approximately equals to the marginal price p^* and which achieve their *global minimum* (*maximum*) over the preference (production) sets at the ε-optimal allocations. Such functions can be interpreted as approximate *nonlinear prices* that support a perturbed *convex-type equilibrium* in nonconvex models.

COROLLARY 4.3. *Under the assumptions of Theorem 4.1 the following hold:*

(i) *Given any* $\varepsilon > 0$ *and any* $\gamma > 0$, *there are* $(x, y, w) \in \prod_{i=1}^{n}[\mathrm{cl}P_i(\bar{x}) \cap (\bar{x}_i + (\varepsilon/2)B] \times \prod_{j=1}^{m}[\mathrm{cl}S_j \cap (\bar{y}_j + (\varepsilon/2)B)] \times \mathrm{cl}W$, $p^* \in E^*\backslash\{0\}$, *and* $\eta > 0$ *such that one has (4.3), (4.4), and*

$$\begin{aligned}&\langle p^*, u_i - x_i\rangle \geqslant -(\varepsilon + \gamma)\|u_i - x_i\|\\ &\quad \text{for all } u_i \in \mathrm{cl}P_i(\bar{x}) \cap (x_i + \eta B), \quad i = 1, \dots, n,\\ &\langle p^*, v_j - y_j\rangle \leqslant (\varepsilon + \gamma)\|v_j - y_j\|\\ &\quad \text{for all } v_j \in \mathrm{cl}S_j \cap (y_j + \eta B), \quad j = 1, \dots, m.\end{aligned}$$

(ii) *Let in addition* E *admit an* $\mathcal{S}$*-smooth bump function from the classes* $\mathcal{S}$ *considered in Proposition 3.3. Then for every* $\varepsilon > 0$ *there are* $(x, y, w) \in \prod_{i=1}^{n}[\mathrm{cl}P_i(\bar{x})$

$\cap(\bar{x}_i + (\varepsilon/2)B)] \times \prod_{j=1}^m [\mathrm{cl}S_j \cap (\bar{y}_j + (\varepsilon/2)B)] \times \mathrm{cl}W$ *and* $p^* \in E^* \setminus \{0\}$ *as well as* $\mathcal{S}$*-smooth functions* $g_i : E \to \mathbb{R}(i = 1, \ldots, n)$ *and* $h_j : E \to \mathbb{R}(j = 1, \ldots, m)$ *such that one has (4.3), (4.4), and*

$$\|\nabla g_i(x_i) - p^*\| \leqslant \varepsilon, i = 1, \ldots, n; \|\nabla h(y_j) - p^*\| \leqslant \varepsilon, j = 1, \ldots, m,$$

where g_i *achieves its global minimum over* $\mathrm{cl}P_i(\bar{x})$ *uniquely at* x_i *for all* $i = 1, \ldots, n$*, and* h_j *achieves its global maximum over* $\mathrm{cl}S_j$ *uniquely at* y_j *for all* $j = 1, ldots, m$*. Moreover, we can choose* g_i *and* h_j *to be convex and concave respectively if* E *admits an equivalent Fréchet smooth norm.*

Proof. To prove (i), we first observe that

$$\hat{N}(\bar{x}; \Omega) + \varepsilon B^* \subset \hat{N}_\varepsilon(\bar{x}; \Omega), \quad \varepsilon \geqslant 0,$$

for any set Ω and then use Proposition 3.3(i) in (4.1) and (4.2). All the assertions in (ii) follow from the assertions (ii) and (iii) of Proposition 3.3 applied to (4.1) and (4.2). □

Now let us derive necessary optimality conditions for Pareto and weak Pareto optimal allocations of $\mathcal{E}$ in the *exact* form of the generalized second welfare theorem. To do it, we need to impose additional compactness assumptions that allow us to pass to the limit in the relations of Theorem 4.1. It happens that the sequential normal compactness of *one* among the preference, production, or net demand constraint sets is sufficient for this purpose.

THEOREM 4.4. *Let* $(\bar{x}, \bar{y})$ *be a Pareto (resp. weak Pareto) locally optimal allocation of the economy* $\mathcal{E}$ *satisfying the corresponding assumptions of Theorem 4.1 with* $\bar{w}$ *defined in (2.2). Assume in addition that either one of th4e sets* $\mathrm{cl}P_i(\bar{x})$, $i = 1, \ldots, n$*, or* $\mathrm{cl}S_j$, $j = 1, \ldots, m$*, or* $\mathrm{cl}W$ *is sequentially normally compact at* $\bar{x}_i$, $\bar{y}_j$*, and* $\bar{w}$ *respectively. Then there is a nonzero price* $p^* \in E^*$ *satisfying*

$$-p^* \in N(\bar{x}_i; \mathrm{cl}P_i(\bar{x})) \text{ for all } i = 1, \ldots, n, \tag{4.16}$$

$$p^* \in N(\bar{y}_j; \mathrm{cl}S_j) \text{ for all } j = 1, \ldots, m, \tag{4.17}$$

$$p^* \in N(\bar{w}; \mathrm{cl}W). \tag{4.18}$$

Proof. Let us prove this theorem by passing to the limit in the relations of Theorem 4.1. Pick an arbitrary sequence $\varepsilon_k \downarrow 0$ as $k \to \infty$ and, according to the latter result, find sequences $(x_k, y_k, w_k) \in \prod_{i=1}^n \mathrm{cl}P_i(\bar{x}) \times \prod_{j=1}^m \mathrm{cl}S_J \times \mathrm{cl}W$ and $p_k^* \in E^*$ satisfying (4.1)–(4.4) with $\varepsilon = \varepsilon_k$ for each $k \in \mathbb{N}$. Obviously $(x_k, y_k, w_k) \to (\bar{x}, \bar{y}, \bar{w})$ as $k \to \infty$. Since E is Asplund and p_k^* are uniformly bounded by (4.4), there is $p^* \in E^*$ such that the sequence $\{p_k^*\}$ converges to p^* in the weak-star topology of E^*. Now passing to the limit in (4.1)–(4.3) as

$k \to \infty$ and taking into account the representation (3.6) of the basic normal cone in Asplund spaces, we arrive at all the relations (4.16)–(4.18).

It remains to prove that $p^* \neq 0$ if one of the sets $\mathrm{cl}P_i(\bar{x})$, $\mathrm{cl}S_j$, and $\mathrm{cl}W$ is sequentially normally compact at the corresponding point. On the contrary, let $p^* = 0$ and assume for definiteness that the set $\mathrm{cl}W$ is sequentially normally compact at $\bar{w}$. Then by (4.3) there is a sequence of $e_k^* \in E^*$ such that

$$p_k^* - \varepsilon_k e_k^* \in \hat{N}(w_k; \mathrm{cl}W) \text{ with } \|e_k^*\| = 1 \text{ for all } k \in \mathbb{N}. \tag{4.19}$$

Obviously $p_k^* - \varepsilon_k e_k^* \xrightarrow{w^*} 0$ as $k \to \infty$. By Definition 3.6 of SNC sets, we conclude from (4.19) that $\|p_k^* - \varepsilon_k e_k^*\| \to 0$ and hence $\|p_k^*\| \to 0$ as $k \to \infty$. The latter contradicts the left-hand inequality in (4.4) for p_k^*. Thus $p^* \neq 0$, which completes the proof of the theorem. □

We can see that Theorem 4.4 requires the sequential normal compactness of *only one* among the preference, production, or net demand constraint sets while the limiting extremal principle of Corollary 3.7 imposes the SNC assumption on *all but one* among these sets. Such an improvement of the general result in the framework of the economic model $\mathcal{E}$ becomes possible mostly due to the *separated structure* of the set (4.7) involved in the extremal system.

Let us present some corollaries of Theorem 4.4 and discuss its relations with other results in this direction. First we consider a special case of $\mathcal{E}$, where the net demand constraint set W admits the representation

$$W = \omega + \Gamma \text{ with some } \omega \in \mathrm{cl}W. \tag{4.20}$$

When $\Gamma = -E_+$ for ordered commodity spaces, representation (4.20) corresponds to the so-called *implicit free disposal of commodities*. We consider a more general case of Γ being an arbitrary *convex cone* in E and show that (4.18) implies in this case the following *complementary slackness condition*, which economically can be interpreted as the *zero value of excess demand* at the marginal price.

COROLLARY 4.5. *In addition to the assumptions of Theorem 4.4, we suppose that W is given as (4.20), where Γ is an nonempty convex subcone of E. Then there is a nonzero price $p^* \in E^*$ satisfying (4.16), (4.17), and*

$$\left\langle p^*, \sum_{i=1}^{n} \bar{x}_i - \sum_{j=1}^{m} \bar{y}_j - \omega \right\rangle = 0. \tag{4.21}$$

Proof. To justify (4.21), we observe that

$$\langle p^*, \bar{w} - \omega \rangle \geqslant \langle p^*, w - \omega \rangle \text{ for all } w \in \mathrm{cl}W \tag{4.22}$$

due to (4.18), (3.4), and (4.20). Hence $\langle p^*, \bar{w} - \omega \rangle \geqslant 0$. On the other hand, taking $2(\bar{w} - \omega) \in W - \omega = \Gamma$ from the cone Γ, we get by (4.22) that $\langle p^*, \bar{w} - \omega \rangle \leqslant 0$, i.e., (4.21) holds. □

In the case of economies with *convex* preference and production sets, relations (4.16) and (4.17) of Theorem 4.4 reduce to the classical *consumer expenditure minimization* and *firm profit maximization* conditions of the second fundamental theorem of welfare economics in Arrow (1951) and Debreu (1951). The corresponding corollary of Theorem 4.4 is formulated as follows.

COROLLARY 4.6. *In addition to the assumptions of Theorem 4.4, we suppose that all the sets* $P_i(\bar{x})$, $i = 1, \ldots, n$, *and* S_j, $j = 1, \ldots, m$, *are convex. Then there is a nonzero price* $p^* \in E^*$ *satisfying (4.18) and such that*

$$\bar{x}_i \text{ minimizes } \langle p^*, x_i \rangle \text{ over } x_i \in \mathrm{cl} P_i(\bar{x}_i) \quad \forall i = 1, \ldots, n, \tag{4.23}$$

$$\bar{y}_j \text{ maximizes } \langle p^*, y_j \rangle \text{ over } y_j \in \mathrm{cl} S_j \quad \forall j = 1, \ldots, m, \tag{4.24}$$

Proof. This follows directly from (4.16) and (4.17) due to the normal cone representation (3.4) for convex sets. □

We have from Proposition 2.4 and the discussion after Definition 3.6 that all the assumptions of Theorem 4.4 automatically hold for weak Pareto (resp. Pareto) optimal allocations if one of the sets $P_i(\bar{x})$, or S_j, or W (resp. $\mathrm{cl} P_i(\bar{x})$) is epi-Lipschitzian at the reference points, which corresponds to their nonempty interiors in the case of convex sets. for *convex* economies with finite-dimensional commodity spaces, the qualification and normal compactness conditions of Theorem 4.4 hold with no interiority assumptions; cf. Debreu (1959) and Cornet (1986). So Corollary 4.6 provides a proper generalization of the classical results of convex welfare economies.

For *nonconvex* economies $\mathcal{E}$, Theorem 4.4 improves the results of Khan (1999) and Cornet (1990) obtained in terms of the normal cone (3.3) for $W = \{\omega\} - \mathbb{R}^n_+$ respectively. Note that, in the nonconvex case, relations (4.16) and (4.17) give *first-order necessary conditions* for consumer's expenditure minimization (4.23) and firm's profit maximization (4.24). Relations of this type are called *marginal pricing quasi-equilibrium* formalized in terms of the corresponding normal cone; cf. Guesnerie (1975) and Cornet (1990).

A version of Theorem 4.4 for nonconvex 'markets clear' economies is presented in Jofré (2000) under the CEL assumption on one of the preference or production sets and an additional robustness (closed graph) subdifferential requirement that holds when E is weakly compactly generated (hence admits a Feéchet smooth renorm) in the framework of Theorem 4.4. Jofré's result also holds in general Banach spaces in terms of bigger Ioffe's and Clarke's normal cones (or the corresponding subdifferentials of the distance function); see the discussions in Section 3. The latter result improves a generalized version of the second welfare theorem obtained in Bonnisseau and Cornet (1988) in terms of Clarke's normal cone under the epi-Lipschitzian property of one of the sets involved. Other extensions of the second welfare theorem for a general economic model with private and public goods are given in Flåm and Jourani (2000) through an abstract subdifferential satisfying full calculus, robustness, and compactness requirements close the ones in to Jofré (2000). These requirements basically restrict the class of subdifferentials and normal cones to those listed above, and they do not cover the general framework of Theorem 4.4. More relaxed requirements are imposed in Mordukhovich (2000b), where the reader can find an abstract version of Theorem 4.4 with further discussions.

Next let us consider a special case of our economic model $\mathcal{E}$ when the commodity space E is an *ordered Banach space* with the *closed positive cone* $E_+ := \{e \in E | e \geqslant 0\}$. The associate closed positive cone E_+^* of the dual space E^* admits the representation

$$E_+^* := \{e^* \in E^* | e^* \geqslant 0\} = \{e^* \in E^* | \langle e^*, e\rangle \geqslant 0 \text{ for all } e \in E_+\}, \tag{4.25}$$

where the order on E^* is induced by the given one $\geqslant$ on E. The last theorem of this paper presents natural conditions ensuring the *positivity* of marginal prices in the framework of Theorem 4.4. The proof is based on the following proposition that exploits specific features of the basic normal cone (3.2) in Banach spaces.

PROPOSITION 4.7. *Let E be an ordered Banach space and $\Omega \subset E$ a nonempty closed subset satisfying the condition*

$$\Omega - E_+ \subset \Omega. \tag{4.26}$$

Then one has

$$N(\bar{e}; \Omega) \subset E_+^* \text{ for every } \bar{e} \in \Omega. \tag{4.27}$$

Proof. Let $e^* \in N(\bar{e}; \Omega)$, where Ω satisfies (4.26). By (3.2) we find sequences $\varepsilon_k \downarrow 0$, $e_k \xrightarrow{\Omega} \bar{e}$, and $e_k^* \xrightarrow{w^*} e^*$ as $k \to \infty$ with $e_k^* \in \hat{N}_{\varepsilon_k}(e_k; \Omega)$ for all $k \in \mathbb{N}$. Due to (4.26) and the obvious *monotonicity* property

$$\hat{N}_\varepsilon(e; \Omega_1) \subset \hat{N}_\varepsilon(e; \Omega) \text{ for any } e \in \Omega_2 \subset \Omega_1 \text{ and } \varepsilon \geqslant 0,$$

we conclude that $e_k^* \in \hat{N}_{\varepsilon_k}(e_k; \Omega - E_+)$ for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and take an arbitrary $\gamma > 0$. Using the definition of ε-normals in (3.1), we find $\eta_k > 0$ so that

$$\langle e_k^*, e - e_k\rangle \leqslant (\varepsilon_k + \gamma)\|e - e_k\| \text{ for all } e \in (e_k + \eta B) \cap (\Omega - E_+). \tag{4.28}$$

It is easy to see that $e_k - \eta_k u \in (e_k + \eta_k B) \cap (\Omega - E_+)$ for any $u \in E_+ \cap B$. Substituting $e := e_k - \eta_k u$ into (4.28), we get

$$\langle e_k^*, -u\rangle \leqslant (\varepsilon_k + \gamma)\|u\| \leqslant \varepsilon_k + \gamma \text{ for all } u \in E_+ \cap B \text{and} k \in \mathbb{N}.$$

Passing to the limit in the latter inequality and taking into account that $e_k^* \xrightarrow{w^*} e^*$ as $k \to \infty$, we arrive at

$$\langle e^*, -u\rangle \leqslant \gamma \text{ for all } u \in E_+ \cap B,$$

which implies $e^* \in E^*_+$ since $\gamma > 0$ was chosen arbitrary. This completes the proof of the proposition. □

It is proved in Jofré and Rivera (1998), using a finite-dimensional technique, that condition (4.26) held for *each* $\bar{x} \in \mathrm{bd}\Omega$ is *necessary and sufficient* for the fulfillment of property (4.27) for closed sets in $\mathbb{R}^n$. Note that (4.26) is related to *free-disposal* type conditions in economic models. The following theorem contains assumptions in this line imposed on either preference, or production, or net demand constraint sets that ensure the price positivity $p^* \in E^*_+\backslash\{0\}$ in our generalized second welfare theorem in the framework or ordered Asplund commodity spaces.

THEOREM 4.8. *Let $(\bar{x}, \bar{y})$ be a Pareto (resp. weak Pareto) locally optimal allocation of the economy $\mathcal{E}$. In addition to the corresponding assumptions of Theorem 4.4, we suppose that E is an ordered space and one of the following conditions holds:*

(a) *There is $i \in \{1, \ldots, n\}$ such that the i-th consumer satisfies the desirability assumption at $\bar{x}$, i.e.,*

$$\mathrm{cl}P_i(\bar{x}) + E_+ \subset \mathrm{cl}P_i(\bar{x})$$

(b) *There is $j \in \{1, \ldots, m\}$ such that the j-th firm satisfies the free-disposal assumption, i.e.,*

$$\mathrm{cl}S_j - E_+ \subset \mathrm{cl}S_j.$$

(c) *The net demand constraint set W exhibits implicit free disposal of commodities, i.e.,*

$$\mathrm{cl}W - E_+ \subset \mathrm{cl}W.$$

Then there is a positive marginal price $p^* \in E^*_+\backslash\{0\}$ satisfying (4.16)–(4.18).

Proof. The marginal price positivity $p^* \in E^*_+$ in cases (b) and (c) follows directly from Proposition 4.7 due to (4.17) and (4.18). Case (a) reduces to the same proposition due to (4.16) and the property

$$N(\bar{e}; \Omega) = -N(-\bar{e}; \Omega) \text{ for every } \Omega \subset E \text{ and } \bar{e} \in \Omega$$

valid in any Banach space. To check this property, it is sufficient to use (3.2) and formula (3.1) for the set of ε-normals. □

Observe that each of the conditions in (a)–(c) implies the epi-Lipschitzian property of the corresponding sets $\mathrm{cl}P_i(\bar{x})$, $\mathrm{cl}S_j$, and $\mathrm{cl}W$ provided that $\mathrm{int}E_+ \neq \emptyset$. Due to the discussions above, the latter assumption ensures also the fulfillment of the qualification and normal compactness conditions of Theorem 4.4 and thus the existence of a positive price $p^* \in E_+\backslash\{0\}$ in Theorem 4.8.

Note that Theorem 4.8 substantially improves, in the Asplund space setting, the main result in Khan (1991) formalized in terms of a bigger Ioffe's normal cone, where $W = \omega - E_+$, *both* conditions (a) and (b) hold for *all* $i = 1, \ldots, n$ and $j = 1, \ldots, m$ respectively, and *every* preference and production set is assumed to be epi-Lipschitzian. Moreover, we do not need a lattice structure of the commodity space, reflexive preference relations, and strong Pareto optimum requirements imposed in that paper.

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References

Arrow, K.J. (1951), An extension of the basic theorems of classical welfare economics, in Neyman, J. (ed.), *Proceedings of the second Berkeley Symposium on Mathematical Statistics and Probability*, pp. 507–532, University of California, Berkeley, California.

Bonnisseau, J.-M. and Cornet, B. (1988), Valuation equilibrium and Pareto optimum, *J. of Mathematical Economics* 17: 293–308.

Borwein, J.M. and Fitzpatrick, S.P. (1995), Weak* sequential compactness and bornological limit derivatives, *J. of Convex Analysis* 2: 59–68

Borwein, J.M. and Jofré, A. (1998), A nonconvex separation property in Banach spaces, *Mathematical Methods of Operations Research* 48: 169–179.

Borwein, J.M., Lucet, Y. and Mordukhovich, B.S. (2000), Compactly epi-Lipschitzian convex sets and functions in normed spaces, *J. of Convex Analysis* 2: 375–394.

Borwein, J.M. and Strojwas, H.M. (1985), Tangential approximations, *Nonlinear Analysis* 9: 1347–1366.

Clarke, F.H. (1983), *Optimization and Nonsmooth Analysis*, Wiley, New York.

Cornet, B. (1986), The second welfare theorem in nonconvex economies, CORE Discussion Paper No. 8630.

Cornet, B. (1990), Marginal cost pricing and Pareto optimality, in Champsaur, P. (ed.), *Essays in Honor of Edmond Malinvaud*, Vol. 1, pp. 14–53, MIT Press, Cambridge, Massachusetts.

Debreu, G. (1951), The coefficient of resource utilization, *Econometrica* 19: 273–292.

Debreu, G. (1959), *Theory of Value*, Wiley, New York.

Dubovitskii, A.Y. and Milyutin, A.A. (1965), Extremum problems in the presence of restrictions, *U.S.S.R. Comp. Maths. Math. Phys.* 5: 1–80.

Fabian, M. and Mordukhovich, B.S. (1998), Nonsmooth characterizations of Asplund spaces and smooth variational principles, *Set-Valued Analysis* 6: 381–406.

Flåm, S.D. and Jourani, A. (2000) Prices and Pareto optima, preprint.

Guesnerie, R. (1975), Pareto optimality in non-convex economies, *Econometrica* 43: 1–29.

Ioffe, A.D. (1989), Approximate subdifferential and applications, III: the metric theory, *Mathematika* 36: 1–38.

Ioffe, A.D. (2000), Codirectional compactness, metric regularity and subdifferential calculus, in Théra, M. (ed.), *Experimental, Constructive, and Nonlinear Analysis*, CMS Conference Proceedings, Vol. 27, pp. 123–164, American Mathematical Society, Providence, Rhode Island.

Jofré, A. (2000), A second welfare theorem in nonconvex economies, in Théra, M. (ed.), *Experimental, Constructive and Nonlinear Analysis*, CMS Conference Proceedings, Vol. 27, pp. 175–184, American Mathematical Society, Providence, Rhode Island.

Jofré, A. and Rivera, J. (1998), The second welfare theorem in a nonconvex nontransitive economy, preprint.

Jofré, A. and Rivera J. (2000), A nonconvex separation property and some applications, to appear in *Mathematical Programming*.

Khan, M.A. (1991), Ioffe's normal cone and the foundations of welfare economics: The infinite dimensional theory, *J. of Mathematical Analysis and Applications* 191: 284–298.

Khan, M.A. (1999), The Mordukhovich normal cone and the foundations of welfare economics, *J. Public Economic Theory* 1: 309–338.

Khan, M.A. and Vohra, R. (1988), Pareto optimal allocations of nonconvex economies in locally convex spaces, *Nonlinear Analysis* 12: 943–950.

Kruger, A.Y. and Mordukhovich, B.S. (1980), Extremal points and the Euler equation in non-smooth optimization, *Doklady Akademii Nauk BSSR* 24: 684–687.

Lange, O. (1942), The foundations of welfare economics, *Econometrica* 10: 215–228.

Mas-Colell, A. (1985), Pareto optima and equilibria: the infinite dimensional case, in Aliprantis, C.D. et al. (eds.), *Advances in Equilibrium Theory*, pp. 25–42, Springer, New York.

Mordukhovich, B.S. (1976), Maximum principle in problems of time optimal control with non-smooth constraints, *J. of Applied Mathematics and Mechanics* 40: 960–969.

Mordukhovich, B.S. (1988), *Approximation Methods in Problems of Optimization and Control*, Nauka, Moscow.

Mordukhovich, B.S. (2000a), The extremal principle and its applications to optimization and economics, in Rubinov, A. and Glover, B.M. (eds.), *Optimization and Related Topics*, Applied Optimization Series, Vol. 47, Kluwer, Dordrecht, pp. 343–369.

Mordukhovich, B.S. (2000b), An abstract extremal principle with applications to welfare economics, *J. of Mathematical Analysis and Applications* 251: 187–216.

Mordukhovich, B.S. and Shao, Y. (1996a), Extremal characterizations of Asplund spaces, *Proceedings of the American Mathematical Society* 124: 197–205.

Mordukhovich, B.S. and Shao, Y. (1996b), Nonsmooth sequential analysis in Asplund spaces, *Transactions of the American Mathematical Society* 348: 1235–1280.

Mordukhovich, B.S. and Shao, Y. (1996c), Nonconvex differential calculus for infinite-dimensional multifunctions, *Set-Valued Analysis* 4: 205–236.

Phelps, R.R. (1993), *Convex Functions, Monotone Operators and Differentiability*, 2nd edition, Springer, Berlin.

Rockafellar, R.T. (1980), Generalized directional derivatives and subgradients of nonconvex functions, *Canadian J. of Mathematics* 32: 257–280.

Rockafellar, R.T. and Wets, R.J.-B. (1998), *Variational Analysis*, Springer, Berlin.

Samuelson, P.A. (1947), *Foundations of Economic Analysis*, Harvard University Press, Cambridge, Massachusetts.